CHAPTER 4

Laplace integrals

A Laplace integral has the form

$$I(x) = \int_a^b f(t) \, e^{x\phi(t)} \, dt$$

(4.1)

where we assume $x > 0$. Typically $x$ is a large parameter and we are interested
in the asymptotic behaviour of $I(x)$ as $x \to +\infty$. Integrating by parts gives

$$I(x) = \frac{1}{x} \int_a^b \frac{f(t)}{\phi'(t)} \cdot \frac{d}{dt} \left( e^{x\phi(t)} \right) \, dt$$

$$= \left[ \frac{1}{x} \cdot \frac{f(t)}{\phi'(t)} \cdot e^{x\phi(t)} \right]_a^b - \frac{1}{x} \int_a^b \frac{d}{dt} \left( \frac{f(t)}{\phi'(t)} \right) \cdot e^{x\phi(t)} \, dt.$$  (4.2)

If the 2nd integral term is asymptotically smaller than the boundary term, i.e.

$$\text{integral term} = o(\text{boundary term}) \quad \text{as} \quad x \to +\infty,$$

then

$$I(x) \sim \frac{1}{x} \cdot \frac{f(b)}{\phi'(b)} \cdot e^{x\phi(b)} - \frac{1}{x} \cdot \frac{f(a)}{\phi'(a)} \cdot e^{x\phi(a)} \quad \text{as} \quad x \to +\infty$$  (4.3)

and we have a useful asymptotic expansion for $I(x)$ as $x \to +\infty$. In general,
this will in fact be the case, i.e. (4.3) is valid, if $\phi(t), \phi'(t)$ and $f(t)$ are con-
tinuous functions (possibly complex) and one of the following three conditions
is satisfied:

1. $\phi'(t) \neq 0$ for $a \leq t \leq b$ and either $f(a) \neq 0$ or $f(b) \neq 0$. These
   conditions ensure that the integral term in (4.2) is bounded and is
   asymptotically smaller than the boundary term;

2. Re $\phi(t) \leq$ Re $\phi(b)$ for $a \leq t \leq b$, Re $\phi'(b) \neq 0$ and $f(b) \neq 0$. These
   conditions do not ensure that the integral term in (4.2) is bounded,
   but by Laplace’s method (see below) they ensure

   $$I(x) \sim \frac{1}{x} \cdot \frac{f(b)}{\phi'(b)} \cdot e^{x\phi(b)} \quad \text{as} \quad x \to +\infty;$$

3. Re $\phi(t) \leq$ Re $\phi(a)$ for $a \leq t \leq b$, Re $\phi'(a) \neq 0$ and $f(a) \neq 0$. Similar
   to the last case, the integral term in (4.2) is not necessarily bounded,
   but by Laplace’s method we can ensure

   $$I(x) \sim -\frac{1}{x} \cdot \frac{f(a)}{\phi'(a)} \cdot e^{x\phi(a)} \quad \text{as} \quad x \to +\infty.$$

Further, if any one of these conditions is met then we may also continue
to integrate by parts to generate further terms in the asymptotic expansion
of $I(x)$; each integration by parts generates a new factor of $1/x$. 

1
4.1. Laplace’s Method

We have seen that for Laplace integrals, integration by parts fails for example, when \( \phi'(t) \) has a zero somewhere in \( a \leq t \leq b \). Laplace’s method is a general technique that allows us to generate an asymptotic expansion for Laplace integrals for large \( x \) in such cases. Recall

\[
I(x) = \int_{a}^{b} f(t) e^{x\phi(t)} \, dt
\]

where we now suppose \( f(t) \) and \( \phi(t) \) are real, continuous functions.

**Basic idea.** If \( \phi(t) \) has a (global) maximum at \( t = c \) with \( a \leq c \leq b \) and if \( f(c) \neq 0 \), then it is only the neighbourhood of \( t = c \) that contributes to the full asymptotic expansion of \( I(x) \) as \( x \to +\infty \). This means that:

**Step 1.** We may approximate \( I(x) \) by \( I(x; \epsilon) \) where

\[
I(x; \epsilon) = \begin{cases} 
\int_{c-\epsilon}^{c+\epsilon} f(t) e^{x\phi(t)} \, dt, & \text{if } a < c < b, \\
\int_{a}^{c+\epsilon} f(t) e^{x\phi(t)} \, dt, & \text{if } c = a, \\
\int_{b-\epsilon}^{b} f(t) e^{x\phi(t)} \, dt, & \text{if } c = b,
\end{cases}
\]

where \( \epsilon > 0 \) is arbitrary, but sufficiently small to guarantee that each of the subranges of integration indicated are contained in the interval \([a, b]\). Such a step is valid if the asymptotic expansion of \( I(x; \epsilon) \) as \( x \to +\infty \) does not depend on \( \epsilon \) and is identical to the asymptotic expansion of \( I(x) \) as \( x \to +\infty \). Both of these results are in fact true since (eg. when \( a < c < b \))

\[
\left| \int_{a}^{c-\epsilon} f(t) e^{x\phi(t)} \, dt \right| + \int_{c+\epsilon}^{b} f(t) e^{x\phi(t)} \, dt
\]

is subdominant to \( I(x) \) as \( x \to +\infty \). This is because \( e^{x\phi(t)} \) is exponentially small compared to \( e^{x\phi(c)} \) for \( a \leq t \leq c - \epsilon \) and \( c + \epsilon \leq t \leq b \). In other words, changing the limits of integration only introduces exponentially small errors (all this can be rigorously proved by integrating by parts). Hence we simply replace \( I(x) \) by the truncated integral, \( I(x; \epsilon) \).

**Step 2.** Now \( \epsilon > 0 \) can be chosen small enough so that (now confined to the narrow region surrounding \( t = c \)) it is valid to replace \( \phi(t) \) by the first few terms in its Taylor series expansion.

- If \( \phi'(c) = 0 \) with \( a \leq c \leq b \) and \( \phi''(c) \neq 0 \), approximate \( \phi(t) \) by
  \[
  \phi(t) \approx \phi(c) + \frac{1}{2} \phi''(c) \cdot (t - c)^2.
  \]
- If \( c = a \) or \( c = b \) and \( \phi'(c) \neq 0 \), approximate \( \phi(t) \) by
  \[
  \phi(t) \approx \phi(c) + \phi'(c) \cdot (t - c).
  \]

In either case approximate \( f(t) \) by the leading order term in its expansion about \( t = c \),

\[
f(t) \approx f(c) \neq 0.
\]

(4.5)
Step 3. Having substituted the approximations for $\phi$ and $f$ indicated above, we now extend the endpoints of integration to infinity, in order to evaluate the resulting integrals (again this only introduces exponentially small errors).

- If $\phi'(c) = 0$ with $a < c < b$, we must have $\phi''(c) < 0$ ($t = c$ is a maximum) and so as $x \to +\infty$
  \[ I(x; \epsilon) \sim \int_{c-\epsilon}^{c+\epsilon} f(c)e^{x(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2)} \, dt \]
  \[ \sim f(c)e^{x\phi(c)} \int_{-\infty}^{\infty} e^{x\phi''(c)(t-c)^2} \, dt \]
  \[ = \frac{\sqrt{2}f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}} \int_{-\infty}^{\infty} e^{-s^2} \, ds \quad (4.6) \]

where in the last step we made the substitution $s^2 = -x\cdot\phi''(c)(t-c)^2$. Since $\int_{-\infty}^{\infty} e^{-s^2} \, ds = \sqrt{\pi}$, we get

\[ I(x; \epsilon) \sim \frac{\sqrt{2\pi f(c)e^{x\phi(c)}}}{\sqrt{-x\phi''(c)}} \quad \text{as} \quad x \to +\infty. \quad (4.7) \]

If $c = a$ or $c = b$, then the leading order behaviour for $I(x)$ is the same as that in (4.7), except multiplied by a factor $\frac{1}{2}$—when we extend the limits of integration, we only do so in one direction, so that the integral in (4.6) only extends over a semi-infinite range.

- If $c = a$ and $\phi'(c) \neq 0$, we must have $\phi'(c) < 0$, and as $x \to +\infty$,
  \[ I(x; \epsilon) \sim \int_{a-\epsilon}^{a+\epsilon} f(a)e^{x(\phi(a)+\phi'(a)(t-a))} \, dt \]
  \[ \sim f(a)e^{x\phi(a)} \int_{-\infty}^{\infty} e^{x\phi'(a)(t-a)} \, dt \]
  \[ = -\frac{f(a)e^{x\phi(a)}}{x\phi'(a)} \]

A similar argument for the case $c = b$, for which $\phi'(b) > 0$, gives

\[ I(x) \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)} \quad \text{as} \quad x \to +\infty. \]

Note. If $\phi(t)$ achieves its global maximum at several points in $[a, b]$, decompose the integral $I(x)$ into several intervals, each containing a single maximum. Perform the analysis above and compare the contributions to the asymptotic behaviour of $I(x)$ (which will be additive) from each subinterval. The final ordering of the asymptotic expansion will then depend on the behaviour of $f(t)$ at the maximal values of $\phi(t)$. If the maximum is such that $\phi'(c) = \phi''(c) = \cdots = \phi^{(m-1)}(c) = 0$ and $\phi^{(m)}(c) \neq 0$ then use the approximation $\phi(t) \approx \phi(c) + \frac{1}{m!}\phi^{(m)}(c) \cdot (t-c)^m$. In (4.5), we assumed $f(c) \neq 0$—see the beginning of this section—the case when $f(c) = 0$ is more delicate and treated in many books (see the bibliography).
Example (Stirling’s formula). We shall try to find the leading order behaviour of the (complete) Gamma function
\[ \Gamma(x + 1) := \int_0^{\infty} e^{-t} t^x \, dt \equiv \int_0^{\infty} e^{-t + x \ln t} \, dt \quad \text{as} \quad x \to +\infty. \]

First we try to convert it more readily to the standard Laplace integral form by making the substitution, \( t = xr \), (this really has the effect of creating the maximum of \( \phi \) away from the origin)
\[
\Gamma(x + 1) = \int_0^{\infty} e^{-xr + x \ln x + x \ln r} r \, dr = x^{x+1} \int_0^{\infty} e^{x(-r + \ln r)} \, dr.
\]
Hence \( f(r) \equiv 1 \) and \( \phi(r) = -r + \ln r \). Since \( \phi'(r) = -1 + \frac{1}{r} \) and \( \phi''(r) = -\frac{1}{r^2} \), for all \( r > 0 \), we conclude that \( \phi \) has a global (and local) maximum at \( r = 1 \).
Hence after collapsing the range of integration to a narrow region surrounding \( r = 1 \), we approximate \( \phi(r) \approx \phi(1) + \phi''(1) \frac{(r - 1)^2}{2} \).

Subsequently extending the range of integration out to infinity again we see that
\[
\Gamma(x + 1) \sim x^{x+1} \int_{-\infty}^{\infty} e^s \cdot e^{-x(r-1)^2} \, ds.
\]
Making the subsitution \( s^2 = \frac{x}{2} \cdot (r - 1)^2 \) and using that \( \int_{-\infty}^{\infty} e^{-s^2} \, ds = \sqrt{\pi} \), then reveals
\[
\Gamma(x + 1) \sim \sqrt{2\pi} x^x \cdot e^{-x} \quad \text{as} \quad x \to +\infty.
\]
When \( x \in \mathbb{N} \), this is Stirling’s formula for the asymptotic behaviour of the factorial function for large integers.

4.2. Watson’s lemma

Based on the ideas above, we can prove a simpler result.

WATSON’S LEMMA. Consider the following example of a Laplace integral
\[
I(x) = \int_0^b f(t) e^{-xt} \, dt \quad (b > 0).
\]
Suppose \( f(t) \) is continuous on \([0, b]\) and has the asymptotic expansion
\[
f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as} \quad t \to 0 + .
\]
We assume that \( \alpha > -1 \) and \( \beta > 0 \) so that the integral is bounded near \( t = 0 \); if \( b = \infty \), we also require that \( f(t) = o(e^{ct}) \) as \( t \to +\infty \) for some \( c > 0 \), to guarantee the integral is bounded for large \( t \). Then
\[
I(x) \sim \sum_{n=0}^{\infty} a_n \Gamma(\alpha + \beta n + 1) \frac{1}{x^{\alpha + \beta n + 1}} \quad \text{as} \quad x \to +\infty. \quad (4.8)
\]

PROOF. Following some of the basic ideas of Laplace’s method outlined above:
4.2. Watson’s Lemma

**Step 1.** Replace $I(x)$ by $I(x; \epsilon)$ where

$$I(x; \epsilon) = \int_0^\epsilon f(t)e^{-xt} \, dt.$$  

(4.9)

This approximation only introduces exponentially small errors for any $\epsilon > 0$.

**Step 2.** We can now choose $\epsilon > 0$ small enough so that the first $N$ terms in the asymptotic series for $f(t)$ are a good approximation to $f(t)$, i.e.

$$\left| f(t) - t^\alpha \sum_{n=0}^N a_n t^{\beta n} \right| \leq K \cdot t^{\alpha + \beta(N+1)},$$  

(4.10)

for $0 \leq t \leq \epsilon$ and $K > 0$ is a constant. Substituting the first $N$ terms in the series for $f(t)$ into (4.9) we see that, using (4.10),

$$\left| I(x; \epsilon) - \sum_{n=0}^N a_n \int_0^\epsilon t^{\alpha + \beta n} e^{-xt} \, dt \right| \leq K \cdot \int_0^\epsilon t^{\alpha + \beta(N+1)} e^{-xt} \, dt$$

$$\leq K \cdot \int_0^\infty t^{\alpha + \beta(N+1)} e^{-xt} \, dt$$

$$= K \cdot \frac{\Gamma(\alpha + \beta + \beta N + 1)}{x^{\alpha + \beta + \beta N + 1}}.$$  

**Step 3.** Replace $\epsilon$ by $\infty$ and use the identity

$$\int_0^\infty t^{\alpha + \beta n} e^{-xt} \, dt \equiv \frac{\Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}}$$

so that

$$I(x) - \sum_{n=0}^N a_n \frac{\Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}} = o(x^{-a-\beta N-1}) \quad \text{as} \quad x \to +\infty.$$  

Since this is true for every $N$, we have proved (4.8) and thus the Lemma. \(\square\)

**Example.** To apply Watson’s lemma to the modified Bessel function

$$K_0(x) := \int_1^\infty (s^2 - 1)^{-\frac{1}{2}} e^{-xs} \, ds,$$

we first substitute $s = t + 1$, so the lower endpoint of integration is $t = 0$:

$$K_0(x) = e^{-x} \int_0^\infty (t^2 + 2t)^{-\frac{1}{2}} e^{-xt} \, dt.$$  

For $|t| < 2$ the binomial theorem implies

$$(t^2 + 2t)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}} \left(1 + \frac{t}{2}\right)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}} \sum_{n=0}^\infty \left(-\frac{t}{2}\right)^n \cdot \frac{\Gamma(n + \frac{1}{2})}{n! \Gamma(\frac{1}{2})}.$$  

Watson’s lemma then immediately tells us that

$$K_0(x) \sim e^{-x} \sum_{n=0}^\infty (-1)^n \cdot \frac{(\Gamma(n + \frac{1}{2}))^2}{2^{n+\frac{1}{2}} n! \Gamma(\frac{1}{2})} x^{n+\frac{1}{2}} \quad \text{as} \quad x \to +\infty.$$
**Note.** We can use Watson’s lemma to determine the leading order behaviour of more general Laplace integrals such as (4.1), as \( x \to +\infty \). Simply make the change of variables \( s = -\phi(t) \) in (4.1) so that

\[
I(x) = - \int_{-\phi(b)}^{-\phi(a)} F(s) e^{-xs} \, ds \quad \text{where} \quad F(s) = - \frac{f(\phi^{-1}(-s))}{\phi'(\phi^{-1}(-s))}.
\]

However, if \( t = \phi^{-1}(-s) \) is intricately *multi-valued*, then use the more direct version of Laplace’s method—for example if \( \phi(t) \) has a global maximum at \( t = c \).